Chair for Algorithms
and Data Structures
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Exercise Sheet 10 — Solutions

Exercise 1 (Hannah)

For $A = \begin{pmatrix} 3 & 5 & 5 \\ 3 & 7 & 1 \end{pmatrix}$, we have $A \cdot A^T = \begin{pmatrix} 59 & 49 \\ 49 & 59 \end{pmatrix}$ and $A^T \cdot A = \begin{pmatrix} 18 & 36 & 18 \\ 36 & 74 & 32 \\ 18 & 32 & 26 \end{pmatrix}$.

For the eigenvalues of $A \cdot A^T$, we have to compute the zeroes of $(59-\lambda)^2 - 49^2 = \lambda^2 - 118 \cdot \lambda + 1080 = (\lambda - 108) \cdot (\lambda - 10)$, hence 108 and 10. The corresponding eigenvectors are easily seen to be $(1,1)^T$ and $(1,-1)^T$.

Without having to compute them, we know that the eigenvalues of $A^T \cdot A$ are 108, 10, and 0. We only need the eigenvectors for 108 and 10, and solving the two corresponding 3×3 systems of equations, we get $(1,2,1)^T$ and $(0,1,-2)^T$, respectively.

Normalizing all these eigenvectors, we get the singular value decomposition

$$\begin{pmatrix} 3 & 5 & 5 \\ 3 & 7 & 1 \end{pmatrix} = \begin{pmatrix} 1/\sqrt{2} & 1\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{pmatrix} \cdot \begin{pmatrix} \sqrt{108} & 0 \\ 0 & \sqrt{10} \end{pmatrix} \cdot \begin{pmatrix} 1/\sqrt{6} & 2/\sqrt{6} & 1/\sqrt{6} \\ 0 & 1/\sqrt{5} & -2/\sqrt{5} \end{pmatrix}.$$

Exercise 2 (Hannah)

We prove the statement in five steps. In the following $|| \cdot ||$ always denotes the Frobenius norm for a matrix and the L_2 norm for a vector (in both cases: square root of the sum of the squares of all entries). Note that for a vector x with entries $x_1, \ldots, x_n$, $||x||^2 = \sum_{i=1}^n x_i^2 = x^T \cdot x$. Also note that the norm of a matrix and of its transpose are equal.

(1) Let U be an $m \times m$ column-orthogonal matrix, that is, $U^T \cdot U = I$. Let x be an $m \times 1$ vector. Then $||U \cdot x||^2 = (U \cdot x)^T \cdot (U \cdot x) = x^T \cdot U^T \cdot U \cdot x = x^T \cdot x = ||x||^2$.

(2) Let again U be an $m \times m$ column-orthogonal matrix, and let A be an $m \times n$ matrix. Let $a_1, \ldots, a_n$ the the n $m \times 1$ columns of A. Then, using (1), we have $||U \cdot A||^2 = ||U \cdot (a_1 \ldots a_n)||^2 = ||(U \cdot a_1 \ldots U \cdot a_n)||^2 = ||U \cdot a_1||^2 + \cdots + ||U \cdot a_n||^2 = ||a_1||^2 + \cdots + ||a_n||^2 = ||(a_1 \ldots a_n)||^2 = ||A||^2$.

(3) Let A be an $m \times n$ matrix with singular value decomposition $U \cdot S \cdot V^T$. Then, using (2) two times, we have $||A||^2 = ||U \cdot S \cdot V^T||^2 = ||S \cdot V^T||^2 = ||(S \cdot V^T)^T||^2 = ||V \cdot S||^2 = ||S||^2$.

(4) Let $A = U \cdot S \cdot V^T$ as above. Let $u_1, \ldots, u_m$ the the m $m \times 1$ column vectors of U, let $s_1, \ldots, s_m$ the the s singular values in decreasing order, and let $v_1, \ldots, v_n$ be the n $n \times 1$ column vectors of V. Then it is easy to see that $A = \sum_{i=1}^m u_i \cdot s_i \cdot v_i^T$.

(5) Let $A = U \cdot S \cdot V^T$ as before. Then by (4), we have $A = \sum_{i=1}^m u_i \cdot s_i \cdot v_i^T$ and $A_k = \sum_{i=1}^k u_i \cdot s_i \cdot v_i^T$,

and hence $A - A_k = \sum_{i=k+1}^{m} u_i \cdot s_i \cdot v_i^T$. Hence the singular value decomposition of $A - A_k$ has $s_{k+1}, \ldots, s_m$ as its singular values, and hence, by (3), we have $||A - A_k||^2 = s_{k+1}^2 + \cdots + s_m^2$.

Exercise 3 (Hannah)

Let $x = \alpha_1 \cdot u_1 + \cdots + \alpha_n \cdot u_n$. By assumption, we have that $\alpha_1 = x \cdot u_1^T \neq 0$. Since u_i is an eigenvector of A with eigenvalue λ_i, we have that $A^k \cdot u_i = \lambda^k \cdot u_i$, and hence $A^k \cdot x = \alpha_1 \cdot \lambda_1^k \cdot u_1 + \cdots \alpha_n \cdot \lambda_n^k \cdot u_n = \alpha_1 \cdot \lambda_1^k \cdot \big(u_1 + (\alpha_2/\alpha_1) \cdot (\lambda_2/\lambda_1)^k \cdot u_2 + \cdots + (\alpha_n/\alpha_1) \cdot (\lambda_n/\lambda_1)^k \cdot u_n\big)$. Since by assumption, the eigenvalues of A are all positive and different and λ_1 is the largest, $\lambda_2/\lambda_1, \ldots, \lambda_n/\lambda_1$ are all < 1. Hence $\lim_{k\to\infty} A^k \cdot x/(\alpha_1 \cdot \lambda_1^k) = u_1$, and hence $\lim_{k\to\infty} A^k \cdot x/||A^k \cdot x|| = u_1$ (it can only converge to a multiple of u_1, and since $A^k \cdot x/||A^k \cdot x||$ has norm 1 for all k, it can only be u_1 itself). As an aside, this also proves that, asymptotically, the norm of $A^k \cdot x$ grows as $\alpha_1 \cdot \lambda_1^k$.

Exercise 4 (Hannah)

Here is a straightforward implementation in C++.

```
#include <vector>
#include <math.h>

int main(int argc, char** argv)
{
  int m = 2;
  std::vector<vector<double> > A;
  A.resize(m + 1);
  for (int i = 1; i <= m; ++i) A[i].resize(m + 1);
  A[1][1] = 2;
  A[2][1] = 1;
  A[1][2] = 1;
  A[2][2] = 2;

  int num_iterations = argc > 1 ? atoi(argv[1]) : 100;
  printf("#iterations = %d\n", num_iterations);
  std::vector<double> x;
  x.resize(m + 1);
  x[1] = 1; x[2] = 2;
  for (int iteration = 1; iteration <= num_iterations; ++iteration)
  {
    std::vector<double> xx(m + 1);
    for (int i = 1; i <= m; ++i)
      for (int j = 1; j <= m; ++j) xx[i] += A[i][j] * x[j];
    double norm = 0;
    for (int i = 1; i <= m; ++i) norm += xx[i] * xx[i];
    norm = sqrt(norm);
    for (int i = 1; i <= m; ++i) x[i] = xx[i] / norm;
  }
  for (int i = 1; i <= m; ++i) printf("%10.5f\n", x[i]);
}
```